Pergamon

Int. J. Solids Structures Vol. 31, No. 5, pp. 613–627, 1994

0020–7683/94 $6.00 + .00

A LINEAR THEORY OF LAMINATED SHELLS ACCOUNTING FOR CONTINUITY OF DISPLACEMENTS AND TRANSVERSE SHEAR STRESSES AT LAYER INTERFACES

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(*Received* 31 *March* 1993; *in revised form* 24 *August* 1993)

Abstract—Based on rigorous kinematical analysis, a unified representation of displacement variation through the thickness of laminated shells is derived in the present paper, and then an approximate theory of laminated shells accounting for continuity of displacements and transverse shear stresses at layer interfaces is established. The governing equations contain only five unknowns.

NOMENCLATURE

V	space occupied by the shell
$V_{(m)}$ $(m = 1, 2, \ldots, k)$	space occupied by the mth layer
$\Omega_{(0)}$	bottom surface of the shell
$\Omega_{(k)}$	top surface of the shell
$\Omega_{(m)}$ $(m = 1, 2, \ldots, k-1)$	interface between the mth and $(m+1)$th layers
A	lateral surface of the shell
h	total thickness of the shell
$\theta^3_{(m)}$ $(m = 1, 2, \ldots, k)$	distance between $\Omega_{(m)}$ and $\Omega_{(0)}$, $\theta^3_{(0)} = 0$ and $\theta^3_{(k)} = h$
$\mathbf{R} = \mathbf{R}(\theta^1, \theta^2)$	position vector of a point (θ^1, θ^2) on $\Omega_{(0)}$
$\mathbf{r} = \mathbf{r}(\theta^1, \theta^2, \theta^3)$	position vector of a point $(\theta^1, \theta^2, \theta^3)$ in V
$\mathbf{a}_\alpha$	covariant base vectors, given by $\mathbf{a}_\alpha = \mathbf{R}_{,\alpha}$
$\mathbf{g}_i$	covariant base vectors, given by $\mathbf{g}_i = \mathbf{r}_{,i}$
$(\ )_{,i}$	differentiation with respect to θ^i
$(\ )\|\|_i$	covariant derivative in V with respect to θ^i
$(\ )\|_\alpha$	covariant derivative on $\Omega_{(0)}$ with respect to θ^α
δ^β_α	mixed Kronecker delta
$\varepsilon_{\alpha\beta}$	covariant components of the two-dimensional permutation tensor
μ^β_α	mixed components of the shifter tensor
b^β_α	mixed components of the curvature tensor
u	determinant of the shifter tensor
t	time.

1. INTRODUCTION

A laminated shell can be regarded as a heterogeneous body consisting of a finite number of homogeneous anisotropic layers bonded together. Investigation on such shells has received great attention in the past three decades, and a variety of laminate shell theories have been presented by many authors. From a theoretical point of view, one of the central issues of various theories is how to account for the effects of transverse deformation. This is because the ratios of the transverse shear modulus to the in-plane modulus in the shells made of advanced composites are much lower than those in the shells made of classical metallic materials, and consequently, the transverse shear deformation plays a more remarkable role in affecting the mechanical behaviours of the laminated shells.

According to different approximations of the displacement field, the existing laminated shell theories could be divided into two kinds, i.e. the global approximation and discrete-layer approximation theories. In the theories of the first kind, the displacement field of a laminated shell is assumed to be continuously and smoothly distributed across the entire thickness, and the laminated shell is actually replaced by an equivalent single-layer anisotropic shell. Examples of these theories are the first-order shear deformation theories based on the linear variation assumption of the displacements in the thickness direction (Dong and

Tso, 1972; Vasilenko, 1987; Palmerio *et al.*, 1990), and the higher-order shear deformation theories based on the nonlinear variation assumption of the displacements in the thickness direction (Whitney and Sun, 1974; Librescu, 1975, 1987; Reddy and Liu, 1985; Stein, 1986; Reissner, 1987; Doxsee, 1989; Touratier, 1992a, b). The advantages of these theories are that the order of the governing equations is independent of the total number of layers, and the theories are adequate in predicting the global responses of the shells, such as deflection, buckling load and vibration frequency, etc. However, the theories do not fulfill the compatibility conditions (i.e. the continuity conditions) of transverse shear stresses at layer interfaces because the strain field is continuous. In the theories of the second kind, piecewise, layer-by-layer displacement or transverse shear stress assumptions through the thickness are introduced. For example, the theories based on piecewise linear approximation for the in-plane displacements and constant transverse displacement in the thickness direction (Zukas and Vinson, 1971; Epstein and Glockner, 1971; Grigolyuk and Chulkov, 1972; Librescu, 1974), and the theories based on piecewise continuous approximation for the transverse shear stresses (Hsu and Wang, 1970; Rath and Das, 1973; Grigolyuk and Kulikov, 1984). Although the discrete-layer approximation theories are very accurate in general, they are quite cumbersome in solving concrete problems because the order of their governing equations depends on the number of layers of the shell. In addition, since, in the theories based on piecewise linear approximation for displacements, the transverse shear stresses are constant within each layer, the compatibility conditions of these stresses are also not satisfied. In view of these reasons, Di Sciuva (1987) has proposed a simplified discrete-layer theory with only five unknowns for describing the deformation of the shells by firstly assuming the in-plane displacements to be piecewise linearly distributed through the thickness, and the transverse displacement to be independent of the thickness coordinate, and then imposing the continuity of transverse shear stresses at layer interfaces. A similar laminated shell theory incorporating the geometrical nonlinearities has been proposed by Librescu and Schmidt (1991). Nevertheless, in their theories the transverse shear stresses are uniform across the entire thickness of the shell, and therefore, the compatibility conditions on the external bounding surfaces are not fulfilled either. In spite of this, the work of Di Sciuva (1987) has provided a good background to the later investigations. Recently, several authors (Savithri and Varadan, 1990, 1993; Di Sciuva, 1991, 1992; Cho and Parmerter, 1992; He, 1993) have modified the displacement assumption in Di Sciuva's theory (1987), and developed some third-order shear deformation plate theories with continuous interlaminar stresses.

The object of this paper is to present a laminated shell theory, which not only satisfies the compatibility conditions of transverse shear stresses both at layer interfaces and on the bounding surfaces, but also is relatively simple for solution. Firstly, an exact unified representation of displacement variation through the thickness of a laminated shell is obtained by rigorous kinematical analysis. To the best knowledge of the author, similar work has not been found in the field literature. Then, by taking certain approximations for the representation of displacement variation and imposing the compatibility conditions of transverse shear stresses at layer interfaces as well as on the bounding surfaces, the number of unknown functions contained in the approximate displacement expressions is reduced to five. The governing equations and the associated boundary conditions, which are in terms of the five unknowns, are obtained by using Hamilton's principle.

2. UNIFIED REPRESENTATION OF DISPLACEMENT VARIATION

Let us consider an undeformed laminated shell consisting of a finite number k of homogeneous anisotropic layers with uniform thickness in a curvilinear coordinate system $\{\theta^i\}$ $(i = 1, 2, 3)$. As shown in Fig. 1, the reference surface $\Omega_{(0)}$, defined by $\theta^3 = 0$, coincides with the bottom surface of the shell, and the θ^3-coordinate-line is perpendicular to $\Omega_{(0)}$. In this paper, the Einsteinian summation convention applies to repeated indices, where Latin indices range from 1 to 3 while Greek indices range from 1 to 2. Let $\Omega_{(k)}$ denote the top surface of the shell and $\Omega_{(m)}$ $(m = 1, 2, \ldots, k-1)$ the interface between the mth and $(m+1)$th layers. Then, the space V occupied by the laminated shell is divided by the $(k-1)$

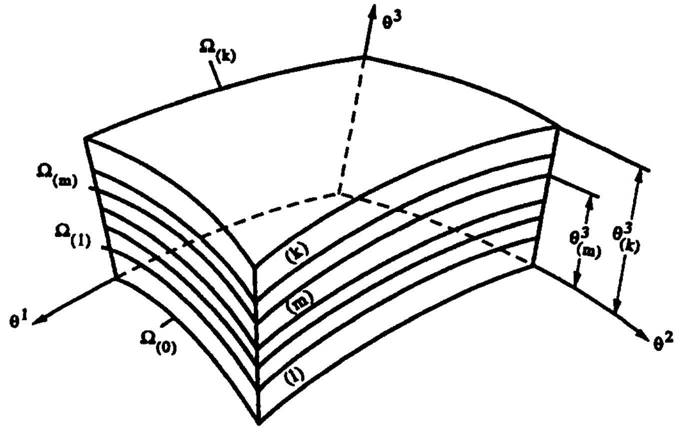

Fig. 1. The geometry of the laminated shell.

surfaces $\Omega_{(m)}$ $(m = 1,2,\ldots,k-1)$ into k subspaces $V_{(m)}$ $(m = 1,2,\ldots,k)$ corresponding to the k layers. The range of $V_{(m)}$ in the θ^3-direction is $[\theta^3_{(m-1)}, \theta^3_{(m)}]$ for $m = 1,2,\ldots,k-1$, and $[\theta^3_{(k-1)}, \theta^3_{(k)}]$ for $m = k$. Here $\theta^3_{(m)}$ $(m = 0,1,\ldots,k)$ is the distance between $\Omega_{(m)}$ and $\Omega_{(0)}$. Obviously, $\theta^3_{(0)} = 0$ and $\theta^3_{(k)} = h$, where h is the total thickness of the shell. After deformation, it is assumed that the displacement vector $\mathbf{v}_{(m)}(\theta^i; t)$ of a point in the mth layer is sufficiently smooth in $V_{(m)}$ in the sense that it is differentiable with respect to θ^i as many times as needed, and can be extended sufficiently smoothly in V. So the displacement vector $\mathbf{v}(\theta^i; t)$ of any point in V can be expressed as

$$\mathbf{v}(\theta^i; t) = \mathbf{v}_{(1)}(\theta^i; t) + \sum_{m=1}^{k-1} [\mathbf{v}_{(m+1)}(\theta^i; t) - \mathbf{v}_{(m)}(\theta^i; t)] H(\theta^3 - \theta^3_{(m)}), \tag{1}$$

where $H(\theta^3 - \theta^3_{(m)})$ is the Heaviside step function,

$$H(\theta^3 - \theta^3_{(m)}) = \begin{cases} 1 & \text{for} \quad \theta^3 \geqslant \theta^3_{(m)}, \\ 0 & \text{for} \quad \theta^3 < \theta^3_{(m)}. \end{cases} \tag{2}$$

Since the vector $\mathbf{v}(\theta^i, t)$ is continuous in V, it requires that

$$\mathbf{v}_{(m+1)}(\theta^\alpha, \theta^3_{(m)}; t) - \mathbf{v}_{(m)}(\theta^\alpha, \theta^3_{(m)}; t) = 0, \quad m = 1,2,\ldots,k-1. \tag{3}$$

By using Taylor expansion and eqn (3), one obtains:

$$\mathbf{v}_{(1)}(\theta^i; t) = \sum_{n=0}^{\infty} \mathbf{u}^{(n)}_{(0)}(\theta^\alpha; t)(\theta^3)^n;$$

$$\mathbf{v}_{(m+1)}(\theta^i; t) - \mathbf{v}_{(m)}(\theta^i; t) = \sum_{n=1}^{\infty} \mathbf{u}^{(n)}_{(m)}(\theta^\alpha; t)(\theta^3 - \theta^3_{(m)})^n, \quad m = 1,2,\ldots,k-1. \tag{4a, b}$$

Thus expression (1) becomes

$$\mathbf{v}(\theta^i; t) = \mathbf{u}^{(0)}_{(0)}(\theta^\alpha; t) + \sum_{m=0}^{k-1} \sum_{n=1}^{\infty} \mathbf{u}^{(n)}_{(m)}(\theta^\alpha; t)(\theta^3 - \theta^3_{(m)})^n H(\theta^3 - \theta^3_{(m)}) \tag{5}$$

in which $\mathbf{u}^{(0)}_{(0)}(\theta^\alpha; t)$ corresponds to the displacement vector of a point on $\Omega_{(0)}$.

Introduce covariant base vectors $\mathbf{a}_i, \mathbf{g}_i$, and contravariant base vectors $\mathbf{a}^i, \mathbf{g}^i$, in the undeformed state of the shell. All of these vectors are defined by

$$\mathbf{a}_\alpha = \mathbf{R}_{,\alpha}, \quad \mathbf{a}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|}, \quad (\mathbf{a}_1 \times \mathbf{a}_2)\cdot\mathbf{a}_3 > 0,$$

$$\mathbf{g}_i = \mathbf{r}_{,i}, \quad (\mathbf{g}_1 \times \mathbf{g}_2)\cdot\mathbf{g}_3 > 0, \quad \mathbf{a}^\alpha\cdot\mathbf{a}_\beta = \delta^\alpha_\beta, \quad \mathbf{a}^3 = \mathbf{a}_3, \quad \mathbf{g}^\alpha\cdot\mathbf{g}_\beta = \delta^\alpha_\beta, \quad \mathbf{g}^3 = \mathbf{g}_3, \tag{6}$$

where $\mathbf{R} = \mathbf{R}(\theta^\nu)$ and $\mathbf{r} = \mathbf{r}(\theta^i)$ denote position vectors of a point on $\Omega_{(0)}$ and a point in V, respectively; $(\ )_{,i}$ denotes differentiation with respect to θ^i, and δ^α_β is the mixed Kronecker delta. Due to $\mathbf{r} = \mathbf{R}+\theta^3\mathbf{a}_3$, there exist the following relations (Naghdi, 1972):

$$\begin{aligned} &\mathbf{g}_\alpha = \mu^\beta_\alpha \mathbf{a}_\beta, \quad \mathbf{g}_3 = \mathbf{a}_3, \quad \mathbf{g}^\alpha = -\mu^{\alpha^{-1}}_\beta \mathbf{a}^\beta, \\ &\mathbf{g}^3 = \mathbf{a}^3, \quad \mathbf{g}_\alpha = \mathbf{g}_{\alpha\beta}\mathbf{g}^\beta, \quad \mathbf{g}^\alpha = g^{\alpha\beta}\mathbf{g}_\beta, \\ &\mathbf{a}_\alpha = a_{\alpha\beta}\mathbf{a}^\beta, \quad \mathbf{a}^\alpha = a^{\alpha\beta}\mathbf{a}_\beta, \end{aligned} \tag{7}$$

in which

$$\begin{aligned} &\mu^\beta_\alpha = \delta^\beta_\alpha - \theta^3 b^\beta_\alpha, \quad \mu^{\alpha^{-1}}_\beta = \frac{1}{\mu}\varepsilon^{\alpha\lambda}\varepsilon_{\beta\nu}\mu^\nu_\lambda, \\ &b^\beta_\alpha = -\mathbf{a}_{3,\alpha}\cdot\mathbf{a}^\beta, \quad g_{\alpha\beta} = \mathbf{g}_\alpha\cdot\mathbf{g}_\beta, \quad g^{\alpha\beta} = \mathbf{g}^\alpha\cdot\mathbf{g}^\beta, \\ &a_{\alpha\beta} = \mathbf{a}_\alpha\cdot\boldsymbol{\alpha}_\beta, \quad a^{\alpha\beta} = \mathbf{a}^\alpha\cdot\mathbf{a}^\beta. \end{aligned} \tag{8}$$

Here $\mu = \det(\mu^\alpha_\beta)$ is the determinant of μ^α_β; $\varepsilon^{\alpha\lambda}$ and $\varepsilon_{\beta\nu}$ are the covariant and contravariant components of the two-dimensional permutation tensor, respectively, which are defined by

$$\varepsilon^{\alpha\lambda} = (\mathbf{a}^\alpha \times \mathbf{a}^\lambda)\cdot\mathbf{a}^3, \quad \varepsilon_{\beta\nu} = (\mathbf{a}_\beta \times \mathbf{a}_\nu)\cdot\mathbf{a}_3. \tag{9}$$

Then, eqn (5) can be written as the following component form with respect to $\mathbf{a}^i$:

$$v_i = u^{(0)}_{(0)i} + \sum_{m=0}^{k-1}\sum_{n=1}^{\infty} u^{(n)}_{(m)i}(\theta^3-\theta^3_{(m)})^n H(\theta^3-\theta^3_{(m)}), \tag{10}$$

where v_i and $u^{(n)}_{m(i)}$ are components of the vector $\mathbf{v}$ and $\mathbf{u}^{(n)}_{(m)}$ with respect to $\mathbf{a}^i$, respectively.

3. APPROXIMATE EXPRESSIONS OF DISPLACEMENTS AND STRAINS

To develop a practical theory of laminated shells, the following approximation of the displacement expression (10) is taken:

$$\begin{aligned} &v_\alpha = u_\alpha + \psi_\alpha\theta^3 + \varphi_\alpha(\theta^3)^2 + \eta_\alpha(\theta^3)^2 + \sum_{m=1}^{k-1} u_{(m)\alpha}(\theta^3-\theta^3_{(m)})H(\theta^3-\theta^3_{(m)}), \\ &v_3 = u_3, \end{aligned} \tag{11a, b}$$

where u_α, u_3, ψ_α, φ_α, η_α and $u_{(m)\alpha}$ replace the quantities $u^{(0)}_{(0)\alpha}$, $u^{(0)}_{(0)3}$, $u^{(1)}_{(0)\alpha}$, $u^{(2)}_{(0)\alpha}$, $u^{(3)}_{(0)\alpha}$ and $u^{(1)}_{(m)\alpha}$ in eqn (10), respectively. The strain components of the shell can be obtained by the formula:

$$e_{ij} = \tfrac{1}{2}(u_{i\|j} + u_{j\|i}), \tag{12}$$

where e_{ij} are the components of strain tensor; $(\ )_{\|j}$ denotes covariant derivative in V with respect to θ^j; and u_i are the components of the displacement vector $\mathbf{v}$ with respect to $\mathbf{g}^i$. For example, by using some of the following relations [see, for example, Naghdi (1972)]:

$$\begin{aligned} &U_\alpha = \mu^\beta_\alpha v_\beta, \quad U_3 = v_3, \quad U^\alpha = \mu^{\alpha^{-1}}_\beta v^\beta, \quad U^3 = v^3, \\ &U_{\alpha\|\beta} = \mu^\nu_\alpha(v_{\nu|\beta} - b_{\nu\beta}v_3), \quad U_{\alpha\|3} = \mu^\nu_\alpha v_{\nu,3}, \\ &U_{3\|\alpha} = v_{3,\alpha} + b^\nu_\alpha v_\nu, \quad U_{3\|3} = v_{3,3}, \end{aligned} \tag{13}$$

and eqns (11a, b), (12), the transverse shear strains are obtained as

$$e_{\alpha 3} = \tfrac{1}{2}(\psi_\alpha + u_{3,\alpha} + b^\beta_\alpha u_\beta) + \varphi_\alpha \theta^3 + \tfrac{1}{2}(3\eta_\alpha - b^\beta_\alpha \varphi_\beta)(\theta^3)^2 - b^\beta_\alpha \eta_\beta (\theta^3)^3 + \frac{1}{2}\sum_{m=1}^{k-1} (\delta^\beta_\alpha - \theta^3_{(m)} b^\beta_\alpha) u_{(m)\beta} H(\theta^3 - \theta^3_{(m)}). \quad (14)$$

Here $(\ )_{|\alpha}$ denotes covariant derivative on $\Omega_{(0)}$ with respect to θ^α. To maintain the uniqueness of the base vectors $\mathbf{g}_i$ at layer interfaces, in the above relations as well as in the following developments all derivatives are stipulated as right-hand derivatives, so that $H_{,3}(\theta^3 - \theta^3_{(m)}) = 0$.

In order to reduce the number of unknowns in eqns (11a, b), the compatibility conditions of transverse shear stresses at layer interfaces and on the bounding surfaces will be used in the following content. For this reason, the constitutive equations of the material of the shell are considered first. As is Librescu (1975), the three-dimensional constitutive equations associated with an elastic anisotropic body may be expressed as

$$\sigma^{\alpha\beta} = H^{\alpha\beta\omega\rho} e_{\omega\rho}, \quad \sigma^{\alpha 3} = 2E^{\alpha 3 \omega 3} e_{\omega 3}, \quad (15a, b)$$

where σ^{ij} are components of stress tensor; E^{ijkl} are components of the elasticity tensor with the symmetry properties $E^{ijkl} = E^{klji} = E^{jikl} = E^{ijlk}$ and $H^{\alpha\beta\omega\rho}$ are defined by

$$H^{\alpha\beta\omega\rho} = E^{\alpha\beta\omega\rho} - \frac{E^{\alpha\beta 33} E^{33\omega\rho}}{E^{3333}}. \quad (16)$$

In eqn (15a, b), the transverse normal stress σ^{33} is ignored, because in the interior region of the shell except the boundary layer (whose width is of the order of the shell thickness) it is much smaller than other stress components.

For the purpose of simplification, assume that no tangential tractions are exerted on $\Omega_{(0)}$ and $\Omega_{(k)}$. With eqn (15b), the compatibility conditions of transverse shear stresses on the bounding surfaces of the shell are written as

$$2E^{\alpha 3\omega 3}_{(1)} e_{\omega 3}(\theta^\beta, 0; t) = 0, \quad 2E^{\alpha 3\omega 3}_{(k)} e_{\omega 3}(\theta^\beta, h; t) = 0, \quad (17a, b)$$

where $E^{ijkl}_{(m)}$ are components of the elasticity tensor associated with the mth layer. Since $\det(E^{\alpha 3\omega 3}) \neq 0$, it is not difficult to obtain the following relations from eqns (14) and (17a, b):

$$\psi_\alpha = -u_{3,\alpha} - b^\beta_\alpha u_\beta,$$
$$\left(b^\beta_\alpha - \frac{3}{2h}\delta^\beta_\alpha\right)\eta_\beta = \frac{1}{h^2}\left(\delta^\beta_\alpha - \frac{h}{2} b^\beta_\alpha\right)\varphi_\beta + \frac{1}{2h^3}\sum_{m=1}^{k-1}(\delta^\beta_\alpha - \theta^3_{(m)} b^\beta_\alpha) u_{(m)\beta}. \quad (18a, b)$$

In eqn (18b), the determinant of the coefficients of η_β is

$$\det\left(b^\beta_\alpha - \frac{3}{2h}\delta^\beta_\alpha\right) = \left(\frac{3}{2h}\right)^2 \hat{\mu},$$

where $\hat{\mu}$ stands for the value of μ at $\theta^3 = \frac{2}{3}h$. Due to $\hat{\mu} \neq 0$ (Naghdi, 1972), eqn (18) gives:

$$\eta_\alpha = d_\alpha^\beta \varphi_\beta + \sum_{m=1}^{k-1} f_{(m)\alpha}^\beta u_{(m)\beta}, \tag{19}$$

in which

$$d_\alpha^\beta = \frac{\varepsilon_{\alpha\sigma}\varepsilon^{\lambda\nu}(2hb_\nu^\sigma - 3\delta_\nu^\sigma)(2\delta_\lambda^\beta - hb_\lambda^\beta)}{9h\hat{\mu}},$$

$$f_{(m)\alpha}^\beta = \frac{\varepsilon_{\alpha\sigma}\varepsilon^{\lambda\nu}(2hb_\nu^\sigma - 3\delta_\nu^\sigma)(\delta_\lambda^\beta - \theta_{(m)}^3 b_\lambda^\beta)}{9h^2\hat{\mu}}. \tag{20a, b}$$

Similarly, with eqn (15b), the compatibility conditions of transverse shear stresses at layer interfaces can be written as

$$2E_{(i+1)}^{\alpha 3\omega 3} e_{\omega 3}(\theta^\alpha, \theta_{(i)}^3; t) = 2E_{(i)}^{\alpha 3\omega 3} \lim_{\varepsilon \to 0^+} e_{\omega 3}(\theta^\beta, \theta_{(i)}^3 - \varepsilon; t), \quad (i = 1, 2, \ldots, k-1). \tag{21}$$

Substituting eqn (14) into (21), and using eqns (18a) and (19), one has:

$$E_{(i)}^{\alpha 3\omega 3}(\delta_\omega^\lambda - \theta_{(i)}^3 b_\omega^\lambda) u_{(i)\lambda} + 2(E_{(i+1)}^{\alpha 3\omega 3} - E_{(i)}^{\alpha 3\omega 3}) \left\{ \sum_{m=1}^{k-1} [\tfrac{3}{2} f_{(m)\omega}^\lambda (\theta_{(i)}^3)^2 - b_\omega^\beta f_{(m)\beta}^\lambda (\theta_{(i)}^3)^3] u_{(m)\lambda} + \frac{1}{2} \sum_{m=1}^{i-1} (\delta_\omega^\lambda - \theta_{(m)}^3 b_\omega^\lambda) u_{(m)\lambda} \right\}$$
$$= 2(E_{(i)}^{\alpha 3\omega 3} - E_{(i+1)}^{\alpha 3\omega 3})[\delta_\omega^\lambda \theta_{(i)}^3 + \tfrac{1}{2}(3d_\omega^\lambda - b_\omega^\lambda)(\theta_{(i)}^3)^2 - b_\omega^\beta d_\beta^\lambda (\theta_{(i)}^3)^3]\varphi_\lambda. \tag{22}$$

In fact, eqn (22) can be regarded as $2(k-1)$ linear algebraic equations of $2(k-1)$ unknowns $u_{(m)\lambda}$ $(m = 1, 2, \ldots, k-1)$. Under this consideration, the following relations between $u_{(i)\lambda}$ and φ_ν can be obtained from eqn (22):

$$u_{(i)\lambda} = a_{(i)\lambda}^\nu \varphi_\nu, \tag{23}$$

where $a_{(i)\lambda}^\nu$ depend only on the curvature of $\Omega_{(0)}$, i.e. b_α^β, and the material properties of various layers. For a given laminated shell b_α^β are determined, therefore, all $a_{(i)\lambda}^\nu$ are known constants. Combining eqn (23) with (19) yields:

$$\eta_\alpha = c_\alpha^\nu \varphi_\nu, \tag{24}$$

where

$$c_\alpha^\nu = d_\alpha^\nu + \sum_{m=1}^{k-1} f_{(m)\alpha}^\beta a_{(m)\beta}^\nu. \tag{25}$$

By substitution of eqns (18a), (23) and (24), the approximate displacement expressions (11a, b) become:

$$v_\alpha = \mu_\alpha^\beta u_\beta - \theta^3 u_{3,\alpha} + h_\alpha^\beta \varphi_\beta, \quad v_3 = u_3, \tag{26a, b}$$

where h_α^β are known functions of θ^3 defined by

$$h_\alpha^\beta = \delta_\alpha^\beta (\theta^3)^2 + c_\alpha^\beta (\theta^3)^3 + \sum_{m=1}^{k-1} a_{(m)\alpha}^\beta (\theta^3 - \theta_{(m)}^3) H(\theta^3 - \theta_{(m)}^3). \tag{27}$$

Then, from eqns (12), (13) and (26a, b), the associated strain components are obtained as

$$e_{\alpha\beta} = \tfrac{1}{2}(\mu_\alpha^\nu \mu_\nu^\lambda|_\beta + \mu_\beta^\nu \mu_\nu^\lambda|_\alpha) u_\lambda + \tfrac{1}{2}\mu_\nu^\lambda(\mu_\alpha^\nu u_\lambda|_\beta + u_\beta^\nu u_\lambda|_\alpha)$$
$$- \tfrac{1}{2}(\mu_\alpha^\lambda b_{\lambda\beta} + \mu_\beta^\lambda b_{\lambda\alpha}) u_3 - \tfrac{1}{2}\theta^3(\mu_\alpha^\nu u_{3,\nu\beta} + \mu_\beta^\nu u_{3,\nu\alpha})$$

$$+\tfrac{1}{2}(\mu_\alpha^\nu h_\nu^\lambda|_\beta+\mu_\beta^\nu h_\nu^\lambda|_\alpha)\varphi_\lambda+\tfrac{1}{2}h_\nu^\lambda(\mu_\alpha^\nu\varphi_\lambda|_\beta+\mu_\beta^\nu\varphi_\lambda|_\alpha),$$

$$e_{\alpha 3}=\tfrac{1}{2}(\mu_\alpha^\nu h_{\nu,3}^\lambda+b_\alpha^\nu h_\nu^\lambda)\varphi_\lambda,\quad e_{33}=0. \tag{28a–c}$$

4. MOTION EQUATIONS AND BOUNDARY CONDITIONS

In the derivation of the motion equations and boundary conditions, use is made of Hamilton's principle:

$$\int_0^t\left(\int_V\sigma^{ij}\delta e_{ij}\,\mathrm{d}V-\int_V\rho\dot{U}^i\delta\dot{U}_i\,\mathrm{d}V-\int_V f^i\delta U_i\,\mathrm{d}V-\int_A s^i\delta U_i\,\mathrm{d}A-\int_\Omega p^i\delta U_i\,\mathrm{d}\Omega\right)\mathrm{d}t=0, \tag{29}$$

where a superposed dot denotes differentiation with respect to time t; ρ is mass density; δ is variational operator; f^i are components of body force; s^i are prescribed components of stress vector per unit area of undeformed lateral surface A of the shell; and p^i are prescribed components of stress vector per unit area of the surface Ω comprising $\Omega_{(0)}$, where $\theta^3=0$, and $\Omega_{(k)}$, where $\theta^3=h$.

By using eqns (26a, b), (28a–c), (13) and (29), assuming that $\dot{\rho}=0$, and performing integration through the thickness of the shell, the following motion equations are obtained:

$$M^{(1)\alpha\beta}|_\beta-N^{(1)\alpha}=I^{(1)\beta\alpha}\ddot{u}_\beta-I^{(2)\beta\alpha}u_{3,\beta}+I^{(3)\beta\alpha}\ddot{\varphi}_\beta-F^{(1)\alpha},$$

$$M^{(2)\alpha\beta}|_{\alpha\beta}+N^{(1)3}=I^{(1)33}\ddot{u}_3+I^{(2)\beta\alpha}|_\beta\ddot{u}_\alpha+I^{(6)\alpha\beta}|_\beta\ddot{\varphi}_\alpha+I^{(2)\beta\alpha}\ddot{u}_\alpha|_\beta$$
$$+I^{(6)\alpha\beta}\ddot{\varphi}_\alpha|_\beta-I^{(4)\alpha\beta}|_\beta\ddot{u}_{3,\alpha}-I^{(4)\alpha\beta}\ddot{u}_{3,\alpha\beta}-P^3-F^{(1)3}-F^{(2)\beta}|_\beta,$$

$$M^{(3)\alpha\beta}|_\beta-N^{(2)\alpha}-N^{(3)\alpha}=I^{(3)\alpha\beta}\ddot{u}_\beta+I^{(5)\beta\alpha}\ddot{\varphi}_\beta-I^{(6)\alpha\beta}\ddot{u}_{3,\beta}-F^{(3)\alpha}. \tag{30a–c}$$

The boundary conditions are:

$$M^{(1)\alpha\beta}n_\beta=S^{(1)\alpha}\quad\text{or}\quad\delta u_\alpha=0,$$
$$[\tfrac{1}{2}(M^{(2)\alpha\beta}+M^{(2)\beta\alpha})|_\alpha+F^{(2)\beta}]n_\beta=S^{(1)3}\quad\text{or}\quad\delta u_3=0,$$
$$M^{(3)\alpha\beta}\eta_\beta=S^{(2)\alpha}\quad\text{or}\quad\delta\varphi_\alpha=0,$$
$$\tfrac{1}{2}(M^{(2)\alpha\beta}+M^{(2)\beta\alpha})n_\beta=S^{(3)\alpha}\quad\text{or}\quad\delta u_{3,\alpha}=0, \tag{31}$$

where

$$[N^{(1)\alpha},N^{(2)\alpha}]=\int_0^h\sigma^{\lambda\beta}\mu_\lambda^\nu[\mu_\nu^\alpha|_\beta,h_\nu^\alpha|_\beta]\mu\,\mathrm{d}\theta^3,$$

$$N^{(3)\alpha}=\int_0^h\sigma^{\lambda 3}(\mu_\lambda^\nu h_{\nu,3}^\alpha+b_\lambda^\nu h_\nu^\alpha)\mu\,\mathrm{d}\theta^3,\quad N^{(1)3}=\int_0^h\sigma^{\alpha\beta}\mu_\alpha^\lambda b_{\lambda\beta}\mu\,\mathrm{d}\theta^3,$$

$$[M^{(1)\alpha\beta},M^{(2)\alpha\beta},M^{(3)\alpha\beta}]=\int_0^h\sigma^{\lambda\beta}\mu_\lambda^\nu[\mu_\nu^\alpha,\theta^3\delta_\nu^\alpha,h_\nu^\alpha]\mu\,\mathrm{d}\theta^3,$$

$$[I^{(1)\alpha\beta},I^{(2)\alpha\beta},I^{(3)\alpha\beta},I^{(4)\alpha\beta},I^{(5)\alpha\beta},I^{(6)\alpha\beta}]$$
$$=\int_0^h\rho a^{\lambda\nu}[\mu_\nu^\alpha\mu_\lambda^\beta,\theta^3\delta_\nu^\alpha\mu_\lambda^\beta,\mu_\lambda^\beta h_\nu^\alpha,(\theta^3)^2\delta_\nu^\alpha\delta_\lambda^\beta,h_\nu^\alpha h_\lambda^\beta,\theta^3\delta_\lambda^\beta h_\nu^\alpha]\mu\,\mathrm{d}\theta^3,$$

$$I^{(1)33}=\int_0^h\rho\mu\,\mathrm{d}\theta^3,$$

$$[F^{(1)\alpha}, F^{(2)\alpha}, F^{(3)\alpha}] = \int_0^h f^\nu \mu_\nu^\beta [\mu_\beta^\alpha, \theta^3 \delta_\beta^\alpha, h_\beta^\alpha] \mu \, d\theta^3, \quad F^{(1)3} = \int_0^h f^3 \mu \, d\theta^3,$$

$$[S^{(1)\alpha}, S^{(2)\alpha}, S^{(3)\alpha}] = \int_0^h s^{\nu\beta} [\mu_\nu^\alpha, h_\nu^\alpha, \theta^3 \delta_\nu^\alpha] n_\beta \mu \, d\theta^3, \quad S^{(1)3} = \int_0^h s^{3\beta} n_\beta \mu \, d\theta^3,$$

$$P^3 = \mu_{(k)} p_{(k)}^3 + p_{(0)}^3, \tag{32}$$

and in addition, s^{ij} are components of s^i with respect to $\mathbf{g}_j$, $\mu_{(k)}$ is the value of μ at $\theta^3 = h$, $p_{(k)}^3$ and $p_{(0)}^3$ are normal loads acting on $\Omega_{(k)}$ and $\Omega_{(0)}$, respectively, and n_β are the components of the outward unit vector normal to the boundary of $\Omega_{(0)}$.

Substituting eqns (15a, b) and (28a, b) into the first six equations in (32) yields:

$$\begin{aligned}
N^{(1)\alpha} &= A^{(1)\alpha\delta} u_\delta + A^{(2)\alpha\delta} \varphi_\delta + B^{(1)\alpha\delta\rho} u_\delta|_\rho - B^{(2)\alpha\delta\rho} u_{3,\delta\rho} + B^{(3)\alpha\delta\rho} \varphi_\delta|_\rho - A^{(1)\alpha 3} u_3, \\
N^{(2)\alpha} &= A^{(2)\alpha\delta} u_\delta + A^{(3)\alpha\delta} \varphi_\delta + B^{(4)\alpha\delta\rho} u_\delta|_\rho - B^{(5)\alpha\delta\rho} u_{3,\delta\rho} + B^{(6)\alpha\delta\rho} \varphi_\delta|_\rho - A^{(2)\alpha 3} u_3, \\
N^{(3)\alpha} &= A^{(4)\alpha\delta} \varphi_\delta, \\
N^{(1)3} &= A^{(1)\delta 3} u_\delta + A^{(2)\delta 3} \varphi_\delta + B^{(1)\delta\rho 3} u_\delta|_\rho - B^{(2)\delta\rho 3} u_{3,\delta\rho} + B^{(3)\delta\rho 3} \varphi_\delta|\rho - A^{(1)33} u_3, \\
M^{(1)\alpha\beta} &= B^{(1)\delta\alpha\beta} u_\delta + B^{(4)\delta\alpha\beta} \varphi_\delta + C^{(1)\alpha\beta\delta\rho} u_\delta|\rho - C^{(2)\alpha\beta\delta\rho} u_{3,\delta\rho} + C^{(3)\alpha\beta\delta\rho} \varphi_\delta|_\rho - B^{(1)\alpha\beta 3} u_3, \\
M^{(2)\alpha\beta} &= B^{(2)\delta\alpha\beta} u_\delta + B^{(5)\delta\alpha\beta} \varphi_\delta + C^{(2)\alpha\beta\delta\rho} u_\delta|_\rho - C^{(4)\alpha\beta\delta\rho} u_{3,\delta\rho} + C^{(5)\alpha\beta\rho 3} \varphi_\delta|_\rho - B^{(2)\alpha\beta 3} u_3, \\
M^{(3)\alpha\beta} &= B^{(3)\delta\alpha\beta} u_\delta + B^{(6)\delta\alpha\beta} \varphi_\delta + C^{(3)\alpha\beta\delta\rho} u_\delta|_\rho - C^{(5)\alpha\beta\delta\rho} u_{3,\delta\rho} + C^{(6)\alpha\beta\delta\rho} \varphi_\delta|_\rho - B^{(3)\alpha\beta 3} u_3,
\end{aligned} \tag{33}$$

where $A^{(n)ij}$, $B^{(n)\alpha\beta j}$ and $C^{(n)\alpha\beta\delta\rho}$ are defined in Appendix A. Then, by substituting eqn (33) into (30a–c) and (31), the motion equations in terms of the five unknowns u_α, u_3 and φ_λ are obtained as

$$\begin{aligned}
& D^{(1)\alpha\delta} u_\delta + D^{(2)\alpha\delta} \varphi_\delta + A^{(1)\alpha 3} u_3 + G^{(1)\alpha\delta\rho} u_\delta|_\rho + G^{(2)\alpha\delta\rho} \varphi_\delta|_\rho - B^{(1)\alpha\rho 3} u_{3,\rho} \\
&\quad + C^{(1)\alpha\beta\delta\rho} u_\delta|_{\rho\beta} + C^{(3)\alpha\beta\delta\rho} \varphi_\delta|_{\rho\beta} - G^{(3)\alpha\delta\rho} u_{3,\delta\rho} - C^{(2)\alpha\beta\delta\rho} u_{3,\delta\rho\beta} \\
&\quad = I^{(1)\beta\alpha} \ddot{u}_{3,\beta} - I^{(2)\beta\alpha} \ddot{u}_{3,\beta} + I^{(3)\beta\alpha} \ddot{\varphi}_\beta - F^{(1)\alpha}, \\
& D^{(1)3\delta} u_\delta + D^{(2)3\delta} \varphi_\delta - D^{(1)33} u_3 + G^{(1)3\delta\rho} u_\delta|_\rho + G^{(2)3\delta\rho} \varphi_\delta|_\rho - D^{(3)3\rho} u_{3,\rho} \\
&\quad + K^{(1)\delta\rho\alpha} u_\delta|_{\rho\alpha} + K^{(2)\delta\rho\alpha\varphi}{}_\delta|_{\rho\alpha} - G^{(3)3\delta\rho} u_{3,\delta\rho} + C^{(2)\delta\rho\alpha\beta} u_\delta|_{\rho\alpha\beta} \\
&\quad + C^{(5)\alpha\beta\delta\rho} \varphi_\delta|_{\rho\alpha\beta} - K^{(3)\delta\rho\alpha} u_{3,\delta\rho\alpha} - C^{(4)\alpha\beta\delta\rho} u_{3,\delta\rho\alpha\beta} \\
&\quad = I^{(1)33} \ddot{u}_3 + I^{(2)\beta\alpha}|_\beta \ddot{u}_\alpha + I^{(6)\alpha\beta}|_\beta \ddot{\varphi}_\alpha + I^{(2)\beta\alpha} \ddot{u}_a|_\beta + I^{(6)\alpha\beta} \ddot{\varphi}_\alpha|_\beta \\
&\quad\quad - I^{(4)\alpha\beta}|_\beta \ddot{u}_{3,\alpha} - I^{(4)\alpha\beta} \ddot{u}_{3,\alpha\beta} - P^3 - F^{(1)3} - F^{(2)\beta}|_\beta, \\
& D^{(3)\alpha\delta} u_\delta + D^{(4)\alpha\delta} \varphi_\delta + A^{(2)\alpha 3} u_3 + G^{(4)\alpha\delta\rho} u_\delta|_\rho + G^{(5)\alpha\delta\rho} \varphi_\delta|_\rho - B^{(3)\alpha\rho 3} u_{3,\rho} \\
&\quad + C^{(3)\delta\rho\alpha\beta} u_{\delta,\rho\beta} + C^{(6)\alpha\beta\delta\rho} \varphi_\delta|_{\rho\beta} - G^{(6)\alpha\delta\rho} u_{3,\delta\rho} - C^{(5)\delta\rho\alpha\beta} u_{3,\delta\rho\beta} \\
&\quad = I^{(3)\alpha\beta} \ddot{u}_\beta + I^{(5)\beta\alpha} \ddot{\varphi}_\beta - I^{(6)\alpha\beta} \ddot{u}_{3,\beta} - F^{(3)\alpha},
\end{aligned} \tag{34a–c}$$

and the boundary conditions are:

$$\begin{aligned}
T^{(1)\alpha} &= S^{(1)\alpha} \quad \text{or} \quad \delta u_\alpha = 0, \\
T^{(1)3} &= S^{(1)3} \quad \text{or} \quad \delta u_3 = 0, \\
T^{(2)\alpha} &= S^{(2)\alpha} \quad \text{or} \quad \delta \varphi_\alpha = 0, \\
T^{(3)\alpha} &= S^{(3)\alpha} \quad \text{or} \quad \delta u_{3,\alpha} = 0,
\end{aligned} \tag{35}$$

where $D^{(n)ij}$, $G^{(n)i\alpha\beta}$ and $K^{(n)\delta\rho\alpha}$ are given in Appendix B, $T^{(n)i}$ are given in Appendix C.

5. NUMERICAL EXAMPLE

In order to assess the accuracy of the present theory, cylindrical bending of cross-ply laminated circular cylindrical shell with simply supported boundary is examined in this section. The exact three-dimensional elasticity solution (3DES) of the same problem has been obtained by Ren (1987). As shown in Fig. 2, the shell has inner radius R and central angle Φ, and is infinite in the θ^1-direction, so it is in a state of plane strain with the θ^2–θ^3 plane. Through simple derivation, it is found:

$$\begin{aligned} &a_{11} = 1, \quad a_{22} = R, \qquad a_{12} = a_{21} = 0, \\ &b_{11} = 1, \quad b_{22} = -R, \qquad b_{12} = b_{21} = 0, \\ &\mu_1^1 = 1, \quad \mu_2^2 = 1+\frac{\theta^3}{R}, \quad \mu_2^1 = \mu_1^2 = 0. \end{aligned} \tag{36}$$

Thus, the displacement components of the shell are:

$$v_1 = 0, \quad v_2 = \mu_2^2 u_2 - \theta^3 u_{3,2} + h_2^2 \varphi_2, \quad v_3 = u_3, \tag{37a–c}$$

and the associated strain components are:

$$\begin{aligned} &e_{22} = \mu_2^2(\mu_2^2 u_{2,2} - \theta^3 u_{3,22} + h_2^2 \varphi_{2,2} - b_{22} u_3), \\ &e_{23} = \tfrac{1}{2}(\mu_2^2 h_{2,3}^2 + b_2^2 h_2^2)\varphi_2, \\ &e_{11} = e_{12} = e_{13} = e_{33} = 0. \end{aligned} \tag{38a–f}$$

From Section 4, the problem considered in this section is deduced to solve the following governing equations:

$$\begin{aligned} &C^{(1)2222}u_{2,22} - C^{(2)2222}u_{3,222} + C^{(3)2222}\varphi_{2,22} - B^{(1)223}u_{3,2} = 0, \\ &C^{(2)2222}u_{2,222} - C^{(4)2222}u_{3,2222} + C^{(5)2222}\varphi_{2,222} - G^{(3)322}u_{3,22} + G^{(1)322}u_{2,2} \\ &\qquad + G^{(2)322}\varphi_{2,2} - D^{(1)33}u_3 = \left(1+\frac{h}{R}\right)p^3, \\ &C^{(3)2222}u_{2,22} - C^{(5)2222}u_{3,222} + C^{(6)2222}\varphi_{2,22} - B^{(3)223}u_{3,2} + D^{(4)22}\varphi_3 = 0, \end{aligned} \tag{39a–c}$$

with the boundary conditions:

$$u_3 = 0, \quad T^{(1)2} = 0, \quad T^{(2)2} = 0, \quad T^{(3)2} = 0, \quad \text{at } \theta^2 = 0 \text{ and } \Phi, \tag{40}$$

where

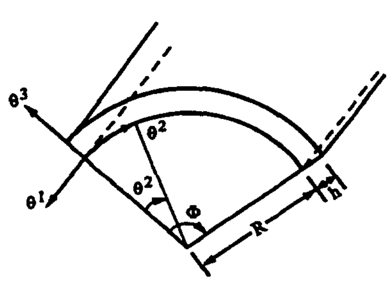

Fig. 2. The laminated circular cylindrical shell.

$$T^{(1)2} = C^{(1)2222}u_{2,2} - C^{(2)2222}u_{3,22} + C^{(3)2222}\varphi_{2,2} - B^{(1)223}u_3,$$
$$T^{(2)2} = C^{(3)2222}u_{2,2} - C^{(5)2222}u_{3,22} + C^{(6)2222}\varphi_{2,2} - B^{(3)223}u_3,$$
$$T^{(3)2} = C^{(2)2222}u_{2,2} - C^{(4)2222}u_{3,22} + C^{(5)2222}\varphi_{2,2} - B^{(2)223}u_3.$$

It is clear that, if $p^3 = p^3(\theta^\alpha)$ can be expressed as

$$p^3 = \sum_{n=1}^{\infty} p^{(n)3} \sin\frac{n\pi}{\Phi}\theta^2, \tag{41}$$

the solutions of eqns (39a–c) are obtained as

$$u_2 = \sum_{n=1}^{\infty} u_2^{(n)} \cos\frac{n\pi}{\Phi}\theta^2, \quad u_3 = \sum_{n=1}^{\infty} u_3^{(n)} \sin\frac{n\pi}{\Phi}\theta^2,$$
$$\varphi_2 = \sum_{n=1}^{\infty} \varphi_2^{(n)} \cos\frac{n\pi}{\Phi}\theta^2, \tag{42a–c}$$

where $u_2^{(n)}$, $u_3^{(n)}$ and $\varphi_2^{(n)}$ are determined by the following linear algebraic equations:

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{bmatrix} u_2^{(n)} \\ u_3^{(n)} \\ \varphi_2^{(n)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \left(1+\frac{h}{R}\right)p^{(n)3} \end{bmatrix}, \tag{43}$$

and the elements k_{ij} are given in Appendix D. With eqns (42a–c), (37a–c), (38a–f) and (15a, b), the displacement, strain and stress components of the shell can be calculated. Of course, all of these components should be finally transformed in physical components, namely,

$$v_{\langle 2\rangle} = \frac{1}{R}v_2, \quad v_{\langle 3\rangle} = v_3, \quad e_{\langle 2\rangle\langle 2\rangle} = \frac{1}{(R+\theta^3)^2}e_{22}, \quad e_{\langle 2\rangle\langle 3\rangle} = \frac{1}{R+\theta^3}e_{23},$$
$$\sigma_{\langle 2\rangle\langle 2\rangle} = (R+\theta^3)^2\sigma^{22}, \quad \sigma_{\langle 2\rangle\langle 3\rangle} = (R+\theta^3)\sigma^{23}. \tag{44}$$

As a special case, numerical computation is performed for a three-layer laminated shell (0°/90°/0°) with $\Phi = \pi/3$. The external load is taken as $p^3 = p_0 \sin 3\theta^2$, and the engineering elastic modulus and Poisson's ratios are given as

$$E_L = 25\times 10^6 \text{ psi } (172 \text{ GPa}), \quad E_T = 10^6 \text{ psi } (6.9 \text{ GPa}),$$
$$G_{LT} = 0.5\times 10^6 \text{ psi } (3.4 \text{ GPa}), \quad G_{TT} = 0.2\times 10^6 \text{ psi } (1.4 \text{ GPa}),$$
$$\nu_{LT} = \nu_{TT} = 0.25, \tag{45}$$

where L denotes the direction parallel to the fibers, and T the transverse direction. The components of the elasticity tensor E^{ijkl} can be obtained from eqn (45) (Jones, 1975). Figure

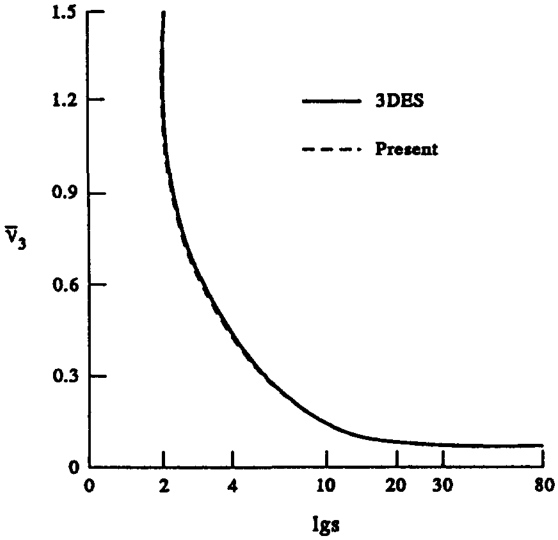

Fig. 3. Relationship between $\bar{v}_3$ and $s(\theta^2 = \pi/3)$.

3 illustrates the relationship between dimensionless deflection $\bar{v}_3$ of the shell and the ratio $s = (R+0.5h)/h$, and Figs 4–6 illustrate the variations of dimensionless in-plane displacement $\bar{v}_2$, in-plane normal stress $\bar{\sigma}_{22}$ and transverse shear stress $\bar{\sigma}_{23}$ through the thickness of the shell for $s = 10$. Here the dimensionless quantities are defined by

$$\bar{v}_3 = \frac{10E_T v_{\langle 3\rangle}}{p_0 h s^4}, \quad \bar{v}_2 = \frac{100E_T v_{\langle 2\rangle}}{p_0 h s^3}, \quad \bar{\sigma}_{22} = \frac{\sigma_{\langle 2\rangle\langle 2\rangle}}{p_0 s^2}, \quad \bar{\sigma}_{23} = \frac{\sigma_{\langle 2\rangle\langle 3\rangle}}{p_0 s}. \tag{46}$$

From the figures one can see that the present results are in good agreement with the 3DES (Ren, 1987), and especially, the transverse shear stress σ^{23} satisfies the compatibility conditions both at layer interfaces and on the bounding surfaces.

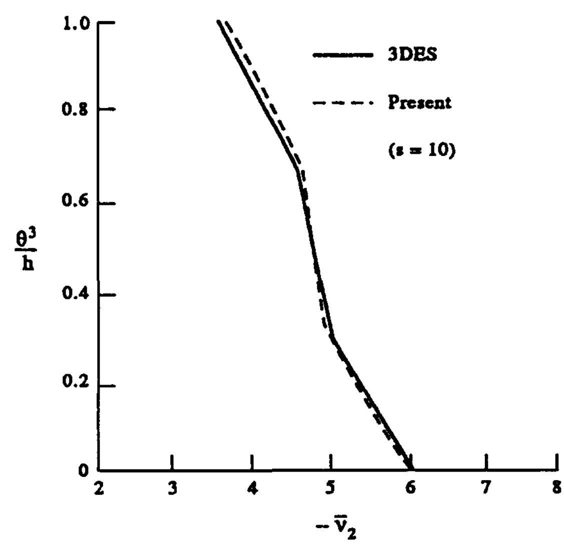

Fig. 4. Variation of $\bar{v}_2$ through thickness ($\theta^2 = \pi/3$).

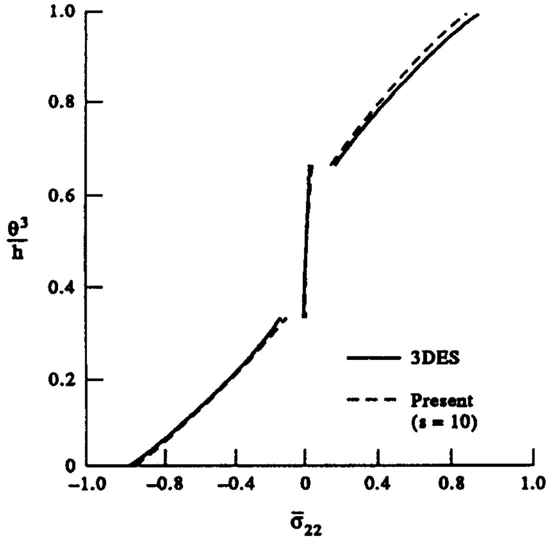

Fig. 5. Variation of $\bar{\sigma}_{22}$ through the thickness ($\theta^2 = \pi/3$).

6. CONCLUSIONS

In this paper, a unified representation of displacement variation through the thickness of a laminated shell has been obtained, which was given in eqn (5). Furthermore, by using the approximate displacement assumption (26a, b) a linear theory of laminated shells accounting for continuity conditions of displacements and transverse shear stresses at layer interfaces as well as the compatability conditions of transverse shear stresses on the bounding surfaces has been established. The governing equations (35a–c) contain only five unknowns: u_α, u_3 and φ_α. The numerical example indicated that the present theory is quite accurate. By setting $b^\beta_\alpha = 0$, a corresponding theory of laminated plates is directly obtained, which is very similar to those proposed by Di Sciuva (1991), Chao and Parmerter (1992); and by setting $\varphi_\alpha = \eta_\alpha = 0$, the approximate displacement assumption (11a, b) will be reduced to that adopted by Di Sciuva (1987), but the latter cannot fulfill the compatibility

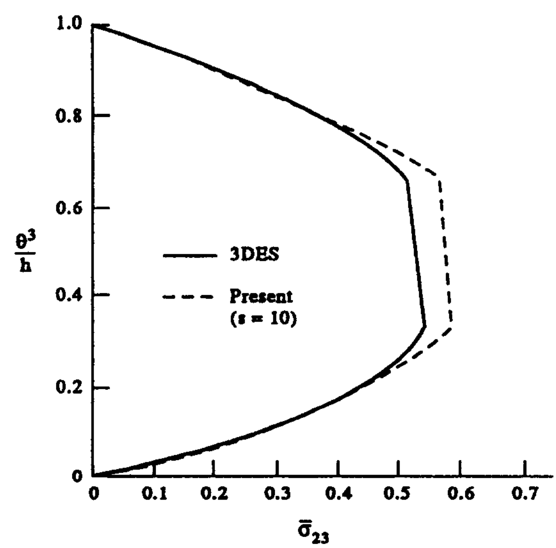

Fig. 6. Variation of $\bar{\sigma}_{23}$ through the thickness ($\theta^2 = \pi/3$).

conditions of transverse shear stresses at layer interfaces and on the bounding surfaces simultaneously.

It is worth noting, the expression (1) is useful in developing laminated shell theories based on displacement assumption. Although in the present paper the displacement representation (5) was obtained by using the Taylor expansions (4a, b), it does not mean that the form of displacement representation must be unique. Actually, different forms of displacement representation can be obtained by using eqn (1) and different expansions which satisfy the condition (3). For example, one can replace eqns (4a, b) by the following Fourier expansions:

$$\mathbf{v}_{(1)}(\theta^i;t) = \mathbf{u}^{(0)}_{(0)}(\theta^\alpha;t)+\sum_{n=1}^{\infty}\left[\mathbf{u}^{(n)}_{(0)s}(\theta^\alpha;t)\sin\frac{n\pi\theta^3}{h}+\mathbf{u}^{(n)}_{(0)c}(\theta^\alpha;t)\cos\frac{n\pi\theta^3}{h}\right],$$

$$\mathbf{v}_{(m+1)}(\theta^i;t)-\mathbf{v}_{(m)}(\theta^i;t) = \sum_{n=1}^{\infty}\mathbf{u}^{(n)}_{(m)}(0^\alpha;t)\sin\frac{n\pi(\theta^3-\theta^3_{(m)})}{h},\quad (m=1,2,\ldots,k-1). \tag{47a, b}$$

Acknowledgement—The author wishes to thank Professors Chien Wei-zang and Liu Ren-huai for their helpful advice.

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APPENDIX A

$$[A^{(1)\alpha\delta}, A^{(2)\alpha\delta}, A^{(3)\alpha\delta}] = \int_0^h H^{\lambda\beta\omega\rho}\mu_\alpha^\sigma\mu_\lambda^\nu[\mu_\sigma^\delta|_\rho\mu_\nu^\alpha|_\beta, \mu_\nu^\alpha|_\beta h_\rho^\delta|_\delta, h_\sigma^\delta|_\beta h_\nu^\alpha|_\beta]\mu\theta^3,$$

$$A^{(4)\alpha\delta} = \int_0^h E^{\lambda 3\omega 3}(\mu_\omega^\sigma h_{\sigma,3}^\delta + b_\omega^\sigma h_\sigma^\delta)(\mu_\lambda^\nu b_{\nu,3}^\alpha + b_\lambda^\nu h_\nu^\alpha)\mu \, \mathrm{d}\theta^3,$$

$$[A^{(1)\alpha 3}, A^{(2)\alpha 3}, A^{(1)33}] = \int_0^h H^{\lambda\beta\omega\rho}\mu_\omega^\sigma\mu_\lambda^\nu b_{\sigma\rho}[\mu_\nu^\alpha|_\beta, h_\nu^\alpha|_\beta, b_{\nu\beta}]\mu \, \mathrm{d}\theta^3,$$

$$[B^{(1)\alpha\delta\rho}, B^{(2)\alpha\delta\rho}, B^{(3)\alpha\delta\rho}, B^{(4)\alpha\delta\rho}, B^{(5)\alpha\delta\rho}, B^{(6)\alpha\delta\rho}]$$

$$= \int_0^h H^{\lambda\beta\omega\rho}\mu_\lambda^\nu[\mu_\sigma^\delta\mu_\omega^\sigma\mu_\nu^\alpha|_\beta, \theta^3\mu_\omega^\delta\mu_\nu^\alpha|_\beta, \mu_\omega^\sigma\mu_\nu^\alpha|_\beta h_\sigma^\delta, \mu_\sigma^\delta\mu_\omega^\sigma h_\nu^\alpha|_\beta, \theta^3\mu_\omega^\delta h_\nu^\alpha|_\beta, \mu_\omega^\sigma h_\sigma^\delta h_\nu^\alpha|_\beta]\mu \, \mathrm{d}\theta^3,$$

$$[B^{(1)\alpha\beta 3}, B^{(2)\alpha\beta 3}, B^{(3)\alpha\beta 3}] = \int_0^h H^{\lambda\beta\omega\rho}\mu_\omega^\sigma b_{\sigma\rho}[\mu_\lambda^\nu\mu_\nu^\alpha, \theta^3\mu_\lambda^\alpha, \mu_\lambda^\nu h_\nu^\alpha]\mu \, \mathrm{d}\theta^3,$$

$$[C^{(1)\alpha\beta\delta\rho}, C^{(2)\alpha\beta\delta\rho}, C^{(3)\alpha\beta\delta\rho}, C^{(4)\alpha\beta\delta\rho}, C^{(5)\alpha\beta\delta\rho}, C^{(6)\alpha\beta\delta\rho}]$$

$$= \int_0^h H^{\lambda\beta\omega\rho}[\mu_\sigma^\delta\mu_\omega^\sigma\mu_\lambda^\nu\mu_\nu^\alpha, \theta^3\mu_\omega^\delta\mu_\lambda^\nu\mu_\nu^\alpha, \mu_\omega^\sigma\mu_\lambda^\nu\mu_\nu^\alpha h_\sigma^\delta, (\theta^3)^2\mu_\omega^\delta\mu_\lambda^\alpha, \theta^3\mu_\omega^\sigma\mu_\lambda^\alpha h_\sigma^\delta, \mu_\omega^\sigma\mu_\lambda^\nu h_\nu^\alpha h_\sigma^\delta]\mu \, \mathrm{d}\theta^3.$$

APPENDIX B

$$D^{(1)\alpha\delta} = B^{(1)\delta\alpha\beta}|_\beta - A^{(1)\alpha\delta}, \quad D^{(2)\alpha\delta} = B^{(4)\delta\alpha\beta}|_\beta - A^{(2)\alpha\delta},$$

$$D^{(3)\alpha\delta} = B^{(3)\delta\alpha\beta}|_\beta - A^{(2)\delta\alpha}, \quad D^{(4)\alpha\delta} = B^{(6)\delta\alpha\beta}|_\beta - A^{(3)\alpha\delta} - A^{(4)\alpha\delta},$$

$$D^{(1)3\delta} = B^{(2)\delta\alpha\beta}|_{\alpha\beta} + A^{(1)\delta 3}, \quad D^{(2)3\delta} = B^{(5)\delta\alpha\beta}|_{\alpha\beta} + A^{(2)\delta 3},$$

$$D^{(3)3\delta} = B^{(2)\alpha\delta 3}|_\alpha + B^{(2)\delta\alpha 3}|_\alpha, \quad D^{(1)33} = B^{(2)\alpha\beta 3}|_{\alpha\beta} + A^{(1)33},$$

$$G^{(1)\alpha\delta\rho} = C^{(1)\alpha\beta\delta\rho}|_\beta + B^{(1)\delta\alpha\rho} - B^{(1)\alpha\delta\rho}, \quad G^{(2)\alpha\delta\rho} = C^{(3)\alpha\beta\delta\rho}|_\beta + B^{(4)\delta\alpha\rho} - B^{(3)\alpha\delta\rho},$$

$$G^{(3)\alpha\delta\rho} = C^{(2)\alpha\beta\delta\rho}|_\beta - B^{(2)\alpha\delta\rho}, \quad G^{(4)\alpha\delta\rho} = C^{(3)\delta\rho\alpha\beta}|_\beta + B^{(3)\delta\alpha\rho} - B^{(4)\alpha\delta\rho},$$

$$G^{(5)\alpha\delta\rho} = C^{(6)\alpha\beta\delta\rho}|_\beta + B^{(6)\delta\alpha\rho} - B^{(6)\alpha\delta\rho}, \quad G^{(6)\alpha\delta\rho} = C^{(5)\delta\rho\alpha\beta}|_\beta - B^{(5)\alpha\delta\rho},$$

$$G^{(1)3\delta\rho} = (B^{(2)\delta\alpha\rho} + B^{(2)\delta\rho\alpha})|_\alpha + C^{(2)\delta\rho\alpha\beta}|_{\alpha\beta} + B^{(1)\delta\rho 3},$$

$$G^{(2)3\delta\rho} = (B^{(5)\delta\alpha\rho} + B^{(5)\delta\rho\alpha})|_\alpha + C^{(5)\alpha\beta\delta\rho}|_{\alpha\beta} + B^{(3)\delta\rho 3},$$

$$G^{(3)3\delta\rho} = C^{(4)\alpha\beta\delta\rho}|_{\alpha\beta} + 2B^{(2)\delta\rho 3}, \quad K^{(1)\delta\rho\alpha} = C^{(2)\delta\rho\alpha\beta}|_\beta + C^{(2)\delta\rho\beta\alpha}|_\beta + B^{(2)\delta\alpha\rho},$$

$$K^{(2)\delta\rho\alpha} = C^{(5)\alpha\beta\delta\rho}|_\beta + C^{(5)\beta\alpha\delta\rho}|_\beta + B^{(5)\delta\alpha\rho}, \quad K^{(3)\delta\rho\alpha} = C^{(4)\beta\alpha\delta\rho}|_\beta + C^{(4)\alpha\beta\delta\rho}|_\beta.$$

APPENDIX C

$$T^{(1)\alpha} = (B^{(1)\delta\alpha\beta}u_\delta + B^{(4)\delta\alpha\beta}\varphi_\delta + C^{(1)\alpha\beta\delta\rho}u_{\delta|\rho} - C^{(2)\alpha\beta\delta\rho}u_{3,\delta\rho} + C^{(3)\alpha\beta\delta\rho}\varphi_{\delta|\rho} - B^{(1)\alpha\beta 3})n_\beta,$$

$$T^{(1)3} = \{[\tfrac{1}{2}(B^{(2)\delta\alpha\beta} + B^{(2)\delta\beta\alpha})u_\delta + \tfrac{1}{2}(B^{(5)\delta\alpha\beta} + B^{(5)\delta\beta\alpha})\varphi_\delta + \tfrac{1}{2}(C^{(2)\delta\rho\alpha\beta} + C^{(2)\delta\rho\beta\alpha})u_{\delta|\rho}$$

$$- \tfrac{1}{2}(C^{(4)\alpha\beta\delta\rho} + C^{(4)\beta\alpha\delta\rho})u_{3,\delta\rho} + \tfrac{1}{2}(C^{(5)\alpha\beta\delta\rho} + C^{(5)\beta\alpha\delta\rho})\varphi_{\delta|\rho} - \tfrac{1}{2}(B^{(2)\alpha\beta 3} + B^{(2)\beta\alpha 3})u_3]|_\alpha + F^{(2)\beta}\}n_\beta,$$

$$T^{(2)\alpha} = (B^{(3)\delta\alpha\beta}u_\delta + B^{(6)\delta\alpha\beta}\varphi_\delta + C^{(3)\delta\rho\alpha\beta}u_{\delta|\rho} - C^{(5)\delta\rho\alpha\beta}u_{3,\delta\rho} + C^{(6)\alpha\beta\delta\rho}\varphi_{\delta|\rho} - B^{(3)\alpha\beta 3}u_3)n_\beta,$$

$$T^{(3)\alpha} = [\tfrac{1}{2}(B^{(2)\delta\alpha\beta} + B^{(2)\delta\beta\alpha})u_\delta + \tfrac{1}{2}(B^{(5)\delta\alpha\beta} + B^{(5)\delta\beta\alpha})\varphi_\delta + \tfrac{1}{2}(C^{(2)\delta\rho\alpha\beta} + C^{(2)\delta\rho\beta\alpha})u_{\delta|\rho}$$

$$- \tfrac{1}{2}(C^{(4)\alpha\beta\delta\rho} + C^{(4)\beta\alpha\delta\rho})u_{3,\delta\rho} + \tfrac{1}{2}(C^{(5)\alpha\beta\delta\rho} + C^{(5)\beta\alpha\delta\rho})\varphi_{\delta|\rho} - \tfrac{1}{2}(B^{(2)\alpha\beta 3} + B^{(2)\beta\alpha 3})u_3]n_\beta.$$

APPENDIX D

$$k_{11}=\frac{n\pi}{\Phi}C^{(1)2222},\quad k_{12}=B^{(1)223}-\left(\frac{n\pi}{\Phi}\right)^2C^{(2)2222},\quad k_{13}=\frac{n\pi}{\Phi}C^{(3)2222},$$

$$k_{21}=\left(\frac{n\pi}{\Phi}\right)^2C^{(3)2222},\quad k_{22}=\frac{n\pi}{\Phi}B^{(3)223}-\left(\frac{n\pi}{\Phi}\right)^3C^{(5)2222},$$

$$k_{23}=\left(\frac{n\pi}{\Phi}\right)^2C^{(6)2222}-D^{(4)22},\quad k_{31}=\left(\frac{n\pi}{\Phi}\right)^3C^{(2)2222}-\frac{n\pi}{\Phi}G^{(1)322},$$

$$k_{32}=-\left(\frac{n\pi}{\Phi}\right)^4C^{(4)2222}+\left(\frac{n\pi}{\Phi}\right)^2G^{(3)322}-D^{(1)33},$$

$$k_{33}=\left(\frac{n\pi}{\Phi}\right)^3C^{(5)2222}-\frac{n\pi}{\Phi}G^{(2)322}.$$